

Two representation theorems of three-valued structures by means of binary relations *

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Abstract: The results here presented are a continuation of the algebraic research line which attempts to find properties of multiple-valued systems based on a poset of two agents.

The aim of this paper is to exhibit two relationships between some three-valued structures and binary relations. The established connections are so narrow that two representation theorems are obtained.

Key Words: Non-functionally complete systems, T -structures, binary relations, three-valued Heyting algebras, rough sets.

1 Introduction

In the domain of reasoning about knowledge, a variety of formalisms have been developed for modelling multi-agent co-operation. In the majority of cases, the set of involved agents is a nonempty set without any structure, the language is a standard modal logic for n agents, and the knowledge of an agent is managed as an epistemic operator.

In order to capture approximation knowledge, an alternative framework to model perception of a group T is provided by n -valued logic. The set of agents is a poset, and the language is based on intuitionistic logic. We have in mind to propose a formalism to express properties of a poset of two co-operating intelligent agents. We intend here to present only algebraic results.

The paper consists of two separate constructions. The first one is motivated by the attempt to represent elements in three-valued structures by pairs of Boolean elements.

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The second construction is motivated by the claim given in [6] that representations using relations are more “natural”.

Both constructions are obtained via a Stone-type representation theorem.

In [13] we considered a three-valued structure which emerged from the formalisation of reasoning with a chain of two agents.

Throughout this paper we will be concerned with an abstract three-valued structure related to Moisil ideas [15], [16], [2] whose definition is given below.

Let (T, \leq) be a chain with $T = \{t_1, t_2\}$ and $t_1 \leq t_2$. In the applications, T can be considered as a poset of two co-operating intelligent agents.

On a distributive lattice $(A, 0, 1, \wedge, \vee)$ with zero and unit we are going to define three unary operators, noted C, S_{t_1}, S_{t_2} . The required properties for these operators are the following:

- the operators S_t , for $t \in \{t_1, t_2\}$, are $(0, 1)$ -lattice homomorphisms from A onto the sublattice $B(A)$ of all complemented elements of A such that $S_t S_w a = S_w a$ for all $t, w \in \{t_1, t_2\}$;
- S_{t_1} and S_{t_2} are respectively an interior and a closure operator on A ([21], pp.115 – 116);
- S_{t_1} is related to the operation C by the equations:
 $S_{t_1} a \wedge C a = 0$ and $S_{t_1} a \vee C a = 1$, for all $a \in A$.

This situation suggests the following definition. For notational convenience, sometimes we replace t_1 and t_2 by their indices (i.e., one and two).

2 T -structures

Definition 2.1 *An abstract algebra $(A, 0, 1, \wedge, \vee, C, S_1, S_2)$ where $0, 1$ are zero-argument operations, C, S_1, S_2 are one argument operations and \wedge, \vee are two-arguments operations is said to be a **Distributive lattice with three unary operators** if*

(T1) $(A, 0, 1, \wedge, \vee)$ is a distributive lattice with zero and unit,

and for every $a, b \in A$ and for all $i, j = 1, 2$, the following equations hold:

(T2) $S_i(a \wedge b) = S_i a \wedge S_i b$; $S_i(a \vee b) = S_i a \vee S_i b$,

(T3) $S_1 a \wedge C a = 0$; $S_1 a \vee C a = 1$,

(T4) $S_i S_j a = S_j a$,

(T5) $S_1 0 = 0$; $S_1 1 = 1$,

(T6) If $S_i a = S_i b$, for all $i = 1, 2$, then $a = b$, (Determination Principle)

(T7) $S_1 a \leq S_2 a$.

This definition is not equational. We will refer to a T -structure A , for short.

Proposition 2.2 *The following properties are true in any T -structure:*

$$(T8) \ S_2 0 = 0 ; S_2 1 = 1,$$

$$(T9) \ a \leq b \text{ if and only if for } i = 1, 2, S_i a \leq S_i b,$$

$$(T10) \ S_1 a \leq a \leq S_2 a,$$

$$(T11) \ S_i a \wedge CS_i a = 0 ; S_i a \vee CS_i a = 1, \text{ for } i = 1, 2.$$

Proof. Indeed by (T5) and (T4) we get $S_2 1 = S_2 S_1 1 = S_1 1 = 1$. The proof of $S_2 0 = 0$ is similar. Thus (T8) holds. Assume $a \leq b$, i.e. $a = a \wedge b$. By (T2), it follows that $S_i a \leq S_i b$. On the other hand, if $S_i a \leq S_i b$, then by (T2), $S_i a = S_i a \wedge S_i b = S_i(a \wedge b)$. Hence by the determination principle (T6), $a = a \wedge b$ and $a \leq b$. Thus (T9) holds. By (T9), the property (T10) is equivalent to $S_i S_1 a \leq S_i a \leq S_i S_2 a$, which is equivalent by (T4) to $S_1 a \leq S_i a \leq S_2 a$. This together with (T7) proves (T10). It follows from (T3) that $S_1 S_i a \wedge CS_i a = 0$ and $S_1 S_i a \vee CS_i a = 1$; by (T4), $S_i a \wedge CS_i a = 0$ and $S_i a \vee CS_i a = 1$. Thus $CS_i a$ is the Boolean complement of $S_i a$, for $i = 1, 2$.

Remark 2.3 *Let $\mathbf{B}(A)$ be the Boolean algebra of all complemented elements in A and $\mathbf{S}_i(A)$ the image of A under S_i , for all $i = 1, 2$. Since by (T2), (T5) and (T8), mappings S_i are $(0, 1)$ -lattice homomorphisms, $S_i(A)$ is a sublattice of A , for all $i = 1, 2$.*

By (T4), $S_i(A) = \{x \in A : S_i x = x\}$ and $\mathbf{S}_1(A) = \mathbf{S}_2(A)$, i.e. mappings S_i have a common image.

By (T11), $S_i(A) \subseteq B(A)$. Using (T2), (T5) and (T10) we get $\mathbf{S}_1(A) = \mathbf{B}(A)$. All these proofs can be found in [10], [11], [14].

By (T11), if “ $-$ ” denotes the Boolean negation we remark that $-S_i a = CS_i a$.

Proposition 2.4 *Let $(A, 0, 1, \wedge, \vee, C, S_1, S_2)$ be a T -structure. We define two operations \Rightarrow and \neg by means of the following equations, for all $a, b \in A$:*

$$a \Rightarrow b = b \vee \bigwedge_{k=1}^2 (CS_k a \vee S_k b), \quad (1)$$

$$\neg a = a \Rightarrow 0. \quad (2)$$

Then the algebra $(A, 0, 1, \wedge, \vee, \Rightarrow, \neg, S_1, S_2)$ is a Heyting algebra with two unary operators satisfying the equation

$$(a \Rightarrow b) \vee (b \Rightarrow a) = 1 \quad (3)$$

that is, a linearly ordered Heyting algebra [18], [19].

Proof. See [13], [9]-[11].

An equivalent **equational** definition of a T -structure is given below.

Definition 2.5 A Heyting algebra with three unary operators (or HT-algebra for short) is an abstract system $A = (A, 0, 1, \wedge, \vee, \Rightarrow, \neg, S_1, S_2)$ such that $0, 1$ are zero-argument operations, \neg, S_1, S_2 are one argument operations and $\wedge, \vee, \Rightarrow$ are two-arguments operations satisfying the following conditions, for all $a, b, c \in A$:

(HT1) $(A, 0, 1, \wedge, \vee, \Rightarrow, \neg)$ is a Heyting algebra,

and for every $a, b \in A$ and for all $i, j = 1, 2$ the following equations hold:

(HT2) $S_i(a \wedge b) = S_i a \wedge S_i b$; $S_i(a \vee b) = S_i a \vee S_i b$,

(HT3) $S_2(a \Rightarrow b) = (S_2 a \Rightarrow S_2 b)$,

(HT4) $S_1(a \Rightarrow b) = (S_1 a \Rightarrow S_1 b) \wedge (S_2 a \Rightarrow S_2 b)$,

(HT5) $S_i S_j a = S_j a$,

(HT6) $S_1 a \vee a = a$,

(HT7) $S_1 a \vee \neg S_1 a = 1$, with $\neg a = a \Rightarrow 0$.

The next two theorems state the equivalence between the notion of T -structure and that of HT-algebra and are proved in [13].

Theorem 2.6 Let $(A, 0, 1, \wedge, \vee, C, S_1, S_2)$ be a T -structure and \Rightarrow and \neg be two operations defined by means of the following equations, for all $a, b \in A$:

$$a \Rightarrow b = b \vee \bigwedge_{k=1}^2 (C S_k a \vee S_k b), \quad (4)$$

$$\neg a = a \Rightarrow 0. \quad (5)$$

Then the algebra $A = (A, 0, 1, \wedge, \vee, \Rightarrow, \neg, S_1, S_2)$ is a HT-algebra.

Conversely:

Theorem 2.7 Let $A = (A, 0, 1, \wedge, \vee, \Rightarrow, \neg, S_1, S_2)$ be a HT-algebra and let us introduce a new operation C by means of the following equation, for all $a \in A$:

$$C a = \neg S_1 a \quad (6)$$

Then the abstract algebra $(A, 0, 1, \wedge, \vee, C, S_1, S_2)$ is a T -structure.

The following general results will be used later on.

Remark 2.8 For a prime filter M in a Heyting algebra, the conditions

(a) M is maximal among the filters which do not contain the element a ,

(b) $a \notin M$ and for every $x \notin M$, $x \Rightarrow a \in M$

are equivalent ([5], p.23).

Remark 2.9 Since S_2 and “ $\neg\neg$ ” are Boolean multiplicative closure operators in the sense of [3], defined on A , it follows that

$$S_2x = \neg\neg x = \bigwedge \{b \in B(A) : x \leq b\}. \quad (7)$$

Two additional facts are recalled for future use. They concern the prime filters in a HT-algebra A and were proved in [13].

Theorem 2.10 *The set of all prime filters in a HT-algebra, ordered by inclusion, is the disjoint union of chains having one or two elements.*

Proposition 2.11 *Let A be a HT-algebra. If P and Q are two prime filters such that $P \subset Q$ and $S_2x \in P \subset Q$ then $x \in Q$.*

3 Examples

For the sake of illustration let us consider some examples depicting the introduced notions. They illustrate our motivations for concrete applications.

1) Let $T = \{t_1, t_2\}$ be an ordered set such that $t_1 \leq t_2$. For each $t \in T$ we denote $F(t)$ the **increasing subset** of T , i.e.

$$F(t) = \{w \in T : t \leq w\}.$$

Let A be the class of the empty set and all increasing sets, i.e.

$$A = \{\emptyset, F(t_2), F(t_1)\}.$$

The class A , ordered by inclusion, is an ordered set with three or two elements, and the system $(A, \emptyset, A, \cap, \cup)$, closed under the operations of intersection and union, is a distributive lattice with zero and unit. For each $t \in T$ we define a special operator S_t on A in the following way:

$$\begin{aligned} S_t(F(x)) &= T \quad \text{if } t \in F(x), \\ S_t(F(x)) &= \emptyset \quad \text{otherwise.} \end{aligned}$$

Finally we define $CF(x) = \neg S_{t_1}(F(x))$. Thus the system $(A, \emptyset, T, \cap, \cup, C, S_{t_1}, S_{t_2})$ is a T -structure, called **basic** T -structure and denoted **BT** or **B** if it has three or two elements respectively.

2) Let Ob be a nonempty set (set of objects) and R an equivalence relation on Ob . Let R^* be the family of all equivalence classes of R , i.e. $R^* = \{x \mid x \in Ob\}$. This family is a partition of Ob . It is well known (see for example [4], [17]) that on the Boolean algebra

$B = (\mathcal{P}(Ob), \emptyset, Ob, \cap, \cup, -)$ where $\mathcal{P}(Ob)$ denotes the powerset of Ob , the equivalence relation R induces a unary operator M in the following way, for $A \subseteq Ob$:

$$MA = \bigcup \{|x| \in R^* : x \in A\};$$

which is equivalent to

$$MA = \bigcup \{|x| \in R^* : |x| \cap A \neq \emptyset\}.$$

By definition we have $M(\emptyset) = \emptyset$ and $A \subseteq MA$. It is well known (see for example [4], [12]) that M also satisfies the condition $M(A \cap MB) = MA \cap MB$, for all $A, B \in \mathcal{P}(Ob)$.

We conclude that M is a monadic operator on the Boolean algebra B and that the system $B = (\mathcal{P}(Ob), \emptyset, Ob, \cap, \cup, -, M)$ is a Monadic Boolean algebra [7], [8]. As usual we define $LA = -M - A$.

Let B^* be the collection of pairs (LA, MA) , where $A \in \mathcal{P}(Ob)$. Since LA and MA are elements of the Boolean algebra $M(\mathcal{P}(Ob))$ of closed elements in B and $LA \subseteq MA$, we consider on B^* the following operations:

$$\begin{aligned} (LA, MA) \wedge (LB, MB) &= (LA \cap LB, MA \cap MB) \\ (LA, MA) \vee (LB, MB) &= (LA \cup LB, MA \cup MB) \\ S_{t_2}(LA, MA) &= (MA, MA) \\ S_{t_1}(LA, MA) &= (LA, LA) \\ C(LA, MA) &= (-LA, -LA) \\ 0 &= (\emptyset, \emptyset) \quad ; \quad 1 = (Ob, Ob) \end{aligned}$$

The right side equalities above are in B^* because the system $(M(\mathcal{P}(Ob)), \emptyset, Ob, \cap, \cup, -, M)$ is a monadic Boolean subalgebra of B .

The system $B^* = (B, 0, 1, \wedge, \vee, C, S_{t_1}, S_{t_2})$ is a T -structure. By the way of example we check the condition (T6). Suppose $S_{t_1}(LA, MA) = S_{t_1}(LB, MB)$ and $S_{t_2}(LA, MA) = S_{t_2}(LB, MB)$ then $(LA, LA) = (LB, LB)$ and $(MA, MA) = (MB, MB)$. We deduce $LA = LB$ and $MA = MB$ and the pairs (LA, MA) and (LB, MB) are equal.

In the literature, a system such as (Ob, R) is called an **approximation space** and a pair (LA, MA) is called a **rough set**. They are concepts related to **Information systems** in the sense of Pawlak [20].

3) Let Ob be a nonempty set and let g be an involution of Ob , i.e. a mapping from Ob into Ob such that $g(g(x)) = x$, for all $x \in Ob$. Clearly, every involution g of Ob is a one-one mapping from Ob onto Ob and $g = g^{-1}$. Let us put for each $X \subseteq Ob$:

$$\begin{aligned} S_1 X &= X \cap g(X) \\ CX &= Ob - (X \cap g(X)) \\ S_2 X &= X \cup g(X). \end{aligned}$$

Let $A(Ob)$ be a nonempty class of subsets of Ob , containing \emptyset and Ob , and closed under set-theoretical intersection and union as well as under the operations C, S_1 and S_2 defined

above. The system $(A(Ob), \emptyset, Ob, \cap, \cup, C, S_1, S_2)$ satisfies the conditions: $(T1), (T3), (T4), (T5), (T7)$ and a half of $(T2)$, namely:

$$\begin{aligned} S_1(X \cap Y) &= S_1X \cap S_1Y \\ S_2(X \cup Y) &= S_2X \cup S_2Y. \end{aligned}$$

Some subalgebras of $(A(Ob), \emptyset, Ob, \cap, \cup, C, S_1, S_2)$ satisfy all the conditions $(T1) - (T7)$, for every $X, Y \subseteq Ob$, i.e. they are **T -algebras of sets**.

These examples are typical, in the sense that every T -structure is isomorphic to a T -structure of sets, as it will be proved in Section 5.

4) If R is a binary relation, we note $R^{-1} = \{(y, x) : (x, y) \in R\}$ the relation inverse. Let E be a nonempty set, ρ a fixed **symmetric** relation on E ($\rho \subseteq E \times E$), and let $(A(E, \rho), \emptyset, \rho, \cap, \cup)$ be a lattice of subsets of ρ .

We can define on $(A(E, \rho), \emptyset, \rho, \cap, \cup)$ the operations S_1, S_2 and C in the following way, for $R \subseteq \rho$:

$$\begin{aligned} S_1(R) &= R \cap R^{-1} \\ S_2(R) &= R \cup R^{-1} \\ C(R) &= \rho - S_1(R). \end{aligned}$$

The system $(A(E, \rho), \emptyset, \rho, \cap, \cup, C, S_1, S_2)$ satisfies the conditions $(T1), (T3), (T4), (T5), (T7)$ and a half of $(T2)$, as in example 3.

Some subalgebras of $(A(E, \rho), \emptyset, \rho, \cap, \cup, C, S_1, S_2)$ satisfy all the conditions $(T1) - (T7)$, for every $X, Y \subseteq E$, i.e. they are **T -algebras of relations**.

4 First construction

In this section we recall the proof of a theorem given in [13], which exhibits a method to construct a concrete T -structure.

Let A be a HT -algebra. By Theorem 2.9, the set Ob of all prime filters in A , ordered by inclusion, is the disjoint union of chains having one or two elements.

Let R_{Ob} be the **binary relation** defined on Ob in the following way:

If $P, Q \in Ob$ then we put $PR_{Ob}Q$ if and only if P and Q are comparable, i.e. if they are in the same chain. R_{Ob} is an **equivalence relation** on Ob .

We consider the Monadic Boolean algebra $(\mathcal{P}(Ob), \emptyset, Ob, \cap, \cup, -, M)$, where for $X \subseteq Ob$:

$$MX = \bigcup \{|P| \in R_{Ob}^* : P \in X\}.$$

Following Stone, for every $x \in A$ we define the map $s : A \rightarrow \mathcal{P}(Ob)$ as follows: $s(x) = \{P \in Ob : x \in P\}$. The map s is a one-one $(0, 1)$ -lattice homomorphism.

Let B^* be the collection of pairs $(Ls(x), Ms(x))$ with operations defined as in example 2. The system $(B^*, \emptyset, Ob, \cap, \cup, C, S_1, S_2)$ is a T -structure. We consider the map $h : A \rightarrow B^*$ defined as follows: $h(x) = (Ls(x), Ms(x))$.

This leads to the result below, showed in [13].

Theorem 4.1 Representation theorem. *Every HT-algebra can be represented as an algebra of rough subsets of an approximation space (Ob, R) .*

5 Second construction

Let A be a HT-algebra and let E be the set of all prime filters in A , ordered by inclusion. According to Theorem 2.9, the ordered set (E, \subseteq) is a disjoint union of chains having one or two elements.

We define the map $g : E \rightarrow E$ in the following way:

$$g(P) = \begin{cases} P, & \text{if } P \text{ is maximal and minimal at the same time,} \\ Q, & \text{if } P \text{ and } Q \text{ are in the same chain and } P \neq Q. \end{cases}$$

The map g is an involution of E . For each $X \subseteq E$ we define the operations S_1, C and S_2 as in example 3. Let $f : A \rightarrow \mathcal{P}(E)$ be the Stone isomorphism, i.e. for each $a \in A$, $f(a) = \{P \in E : a \in P\}$. It is well know that f is a one-one $(0, 1)$ -lattice homomorphism.

We show that f satisfies also the conditions:

$$f(S_1a) = S_1f(a), \quad f(S_2a) = S_2f(a), \quad f(Ca) = Cf(a).$$

By the way of example we show the condition $f(S_2a) = S_2f(a)$. The proof of this equality is accomplished in four steps:

- (i) $f(S_2a) \subseteq S_2(f(S_2a))$. Immediate from the definition of S_2 .
- (ii) $S_2(f(S_2a)) \subseteq f(S_2a)$. Assume $P \in S_2(f(S_2a)) = f(S_2a) \cup g(f(S_2a))$. If $P \in f(S_2a)$ then $S_2a \in P$ and the result is true. If $P \in g(f(S_2a))$ then $S_2a \in g(P)$; if $S_2a \notin P$ then $P \subset g(P)$. In this case, $\neg S_2a \in P$ and $S_2a \wedge \neg S_2a = 0 \in g(P)$, a contradiction.
- (iii) $f(S_2a) \subseteq S_2f(a)$. Let $P \in f(S_2a)$, i.e. $S_2a \in P$. We show that $a \in P$ or $a \in g(P)$. We distinguish three cases. Assume P is minimal and $P \subset g(P)$; by Proposition 2.11 we have $a \in g(P)$. Assume $g(P) \subset P$. Since $S_2a \vee \neg S_2a = 1 \in g(P)$ we deduce either $S_2a \in g(P)$ or $\neg S_2a \in g(P)$. If $\neg S_2a \in g(P)$ we would have $\neg S_2a \wedge S_2a = 0 \in P$ which is impossible, so $S_2a \in g(P)$. By Proposition 2.11 again, we get $a \in P$. Assume P is minimal and maximal, then $P = g(P)$. If $a \notin P$ then by Remark 2.8 we have $\neg a \in P$, and by Remark 2.9 it follows that $\neg a \wedge S_2a = \neg a \wedge \neg \neg a = 0 \in P$ a contradiction. We have shown that either $P \in f(a)$ or $g(P) \in f(a)$. In both cases we conclude $P \in f(a) \cup g(f(a)) = S_2f(a)$.
- (iv) $S_2f(a) \subseteq f(S_2a)$. Since $a \leq S_2a$ then $f(a) \subseteq f(S_2a)$ and $S_2f(a) \subseteq S_2f(S_2a) = f(S_2a)$ by (i) and (ii) above.

The image $f(A)$ is a **T -algebra of sets**.

The set $G = \{(P, g(P))\}_{P \in E}$ is a **symmetric relation** on E .

We consider the map $h : A \rightarrow G \cap (\{f(a)\}_{a \in A} \times E)$ defined by:

$$h(a) = G \cap (f(a) \times E).$$

This map h preserves all the operations on A . In fact :

1. h is one-one.

In fact, if $a \neq b$ then $f(a) \neq f(b)$ (Stone). Suppose without loss of generality that $x \in f(a)$ and $x \notin f(b)$ then $(x, g(x)) \in G \cap (f(a) \times E)$, but $(x, g(x)) \notin G \cap (f(b) \times E)$; thus $h(a) \neq h(b)$ as desired.

2. $h(a \wedge b) = h(a) \cap h(b)$, $h(a \vee b) = h(a) \cup h(b)$.

$$\begin{aligned} h(a \wedge b) &= G \cap (f(a \wedge b) \times E) = G \cap (f(a) \cap f(b) \times E) = \\ &= G \cap [f(a) \times E] \cap [f(b) \times E] = G \cap (f(a) \times E) \cap G \cap (f(b) \times E) = h(a) \cap h(b). \end{aligned}$$

The proof of the other equality is similar.

3. $h(S_2 a) = S_2 h(a)$, $h(S_1 a) = S_1 h(a)$, $h(Ca) = Ch(a)$.

$$\begin{aligned} h(S_2 a) &= G \cap (f(S_2 a) \times E) = G \cap (S_2 f(a) \times E) = G \cap ((f(a) \cup g(f(a))) \times E) = \\ &= [G \cap (f(a) \times E)] \cup [G \cap (g(f(a)) \times E)]. \text{ On the other hand,} \\ S_2 h(a) &= h(a) \cup (h(a))^{-1} = [G \cap (f(a) \times E)] \cup [G \cap (f(a) \times E)]^{-1} = \\ &= [G \cap (f(a) \times E)] \cup [G^{-1} \cap (f(a) \times E)^{-1}] = [G \cap (f(a) \times E)] \cup [G \cap (E \times f(a))]. \end{aligned}$$

We show that $[G \cap (g(f(a)) \times E)] = [G \cap (E \times f(a))]$. In fact, it is a consequence of the following equivalent conditions:

$$(x, y) \in G \cap (g(f(a)) \times E) \Leftrightarrow (x, y) \in G, y = g(x) \text{ and } x \in g(f(a)) \Leftrightarrow (x, y) \in G, g(x) = y \in f(a) \Leftrightarrow (x, y) \in G \cap (E \times f(a)).$$

The proof of the other two equalities are similar.

Remark 5.1 *The operation S_2 defined above, satisfies the following inequalities, for $R, S \subseteq \rho$:*

$$\begin{aligned} S_2(R \cap S) &\subseteq S_2 R \cap S_2 S, \\ S_1(R \cup S) &\supseteq S_1 R \cup S_1 S. \end{aligned}$$

In fact, $S_2(R \cap S) = (R \cap S) \cup (R \cap S)^{-1} = (R \cap S) \cup (R^{-1} \cap S^{-1})$ and $S_2(R) \cap S_2(S) = (R \cup R^{-1}) \cap (S \cup S^{-1}) = (R \cap S) \cup (R \cap S^{-1}) \cup (R^{-1} \cap S) \cup (R^{-1} \cap S^{-1})$.

In the other case the proof is similar.

In general, the equalities are not true, i.e. the system is not a T -structure. Nevertheless, some subalgebras of this system may be. We close the paper with the following result.

We claim that the h -image $(G \cap (\{f(a)\}_{a \in A} \times E), \emptyset, G, \cap, \cup, C, S_1, S_2)$ of A is a T -structure of relations isomorphic to A .

By the way of example we show one of the conditions in (T2):

$$S_2(h(a) \cap h(b)) = S_2(h(a)) \cap S_2(h(b)).$$

In fact, taking into account the condition (T2) in A and the fact that h is a homomorphism, we get:

$$\begin{aligned} S_2(h(a) \cap h(b)) &= S_2(h(a \wedge b)) = h(S_2(a \wedge b)) = h(S_2(a) \wedge S_2(b)) = \\ &= h(S_2(a)) \cap h(S_2(b)) = S_2(h(a)) \cap S_2(h(b)). \end{aligned}$$

This completes the proof of the following statement.

Theorem 5.2 Representation theorem.

Every T -structure $(A, 0, 1, \wedge, \vee, C, S_1, S_2)$ is isomorphic to a T -structure of relations.

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